

Matched Filters for Binary Signals

S. Zohar

Communications Systems Research Section

Matched filters for the optimal high-speed detection of binary signals are designed and their performance as a function of their complexity is explored. The range of filters designed extends from a 2-element filter whose performance is about 0.7 dB below the ideal filter, up to a 20-element filter with a degradation of about 0.1 dB.

I. Introduction

Let $x(t)$ be a binary signal with the levels ± 1 and let the k th bit (a_k) be transmitted during the interval

$$(k-1)T \leq t < kT \quad (1)$$

In the absence of noise, sampling $x(t)$ at any point in the interval (1) will yield a_k . With noise present, the search for an optimal detector leads to the idea of a matched receiver, that is, a receiver whose transfer function is tailored to optimize the detection of a signal having a prescribed spectrum (Ref. 1).

In the simple case considered here, the properties of the matched receiver can be obtained directly without resorting to the general theorem. We adopt this approach here in the interest of completeness.

To determine bit a_k in the presence of noise, we average the signal over the k th bit period

$$\bar{x}_k = \frac{1}{T} \int_{(k-1)T}^{kT} x(t) dt \quad (2)$$

The average \bar{x}_k is now used to determine a_k according to the following decision rule:

$$\left. \begin{aligned} a_k &= 1 \text{ if } \bar{x}_k \geq 0 \\ a_k &= -1 \text{ if } \bar{x}_k < 0 \end{aligned} \right\} \quad (3)$$

A straightforward realization of this scheme would call for an integrator that would have to be sampled and reset at the bit rate $1/T$.

Alternatively, recalling that the output of a linear filter is a convolution of its input, it is conceivable that a special type of filter can be designed such that when $x(t)$ is its input, its output $y(t)$ approximates

$$y(kT) \approx \bar{x}_k \quad (4)$$

Thus, the integrator would be replaced by a presumably cheaper passive filter. The main advantage, however, would be in the elimination of periodic resetting. The conceived filter would operate continuously. Sampling its output at $t = kT$ and applying the decision rule in Eq. (3) would yield a_k .

II. The Matched Filter

Our goal in this section is to show that a filter satisfying Eq. (4) is indeed possible.

Anticipating that \bar{x}_k will be proved to be a sample of the filter output, we seek to rephrase Eq. (2) as a convolution. This calls for the "box-car" function $B_T(t)$ defined in terms of the step function $u(t)$ as follows:

$$B_T(t) = \frac{1}{T} [u(t) - u(t - T)] \quad (5)$$

Using Eq. (5), Eq. (2) can be rephrased as follows:

$$\bar{x}_k = \int_0^\infty x(t) B_T[t - (k-1)T] dt \quad (6)$$

Noting now that $B_T(t)$ satisfies

$$B_T(t + T) = B_T(-t) \quad (7)$$

we get the final form

$$\bar{x}_k = \int_0^\infty x(t) B_T(kT - t) dt \quad (8)$$

The convolution on the right can be interpreted as the output (at time kT) of a filter fed by $x(t)$ and having the impulse response $B_T(t)$.

Denoting the filter output as $y(t)$, Eq. (8) states

$$\bar{x}_k = y(kT) \quad (9)$$

Strictly speaking, such a filter is not realizable in terms of lumped parameters. However, a sufficiently good approximation to it is realizable.

One such realization which approximates $B_T(t)$ while minimizing the integral of the squared error has been described (Ref. 2). It can be shown, however, that the telemetry detection performance of any approximation also depends on a sequence of integrals of the first power of the error. Thus, while elaborate optimization techniques have been used to obtain the networks of (Ref. 2), they are not optimal for the present application.

We shall see that a simple straightforward approximation method due to Guillemin (Ref. 3) yields reasonably good results. For certain parameter values, these are better than the results obtained with the networks of Ref. 2. Guillemin's method is illustrated in Fig. 1. Let $H(s)$, $H_1(s)$, $H_2(s)$ be the Laplace transforms of $h(t)$, $h_1(t)$, $h_2(t)$, respectively. Then $H(s)$, the desired transfer

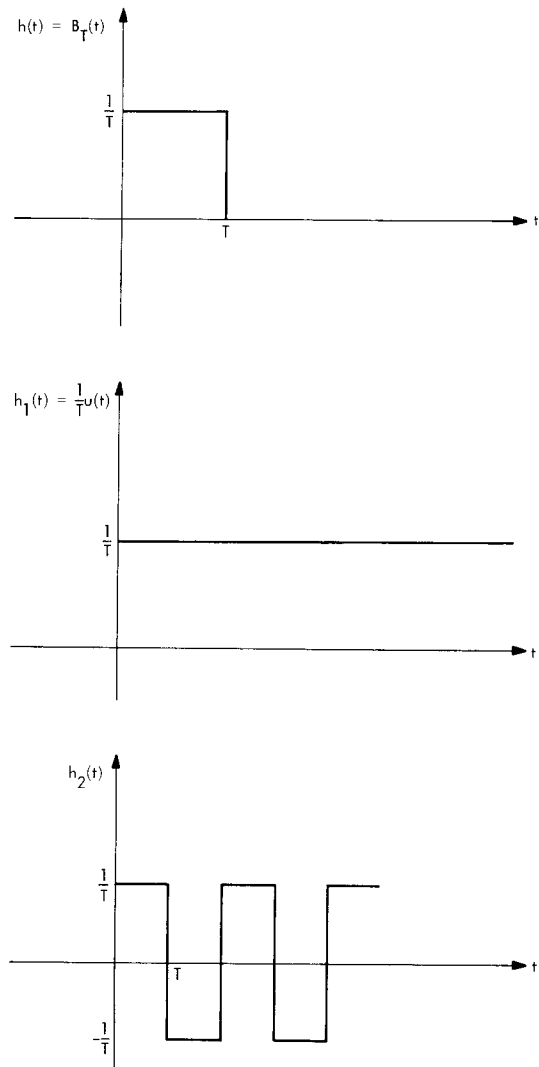


Fig. 1. Functions generating $B_T(t)$

function, can be readily expressed in terms of $H_1(s) = 1/sT$, using the shift operator e^{-sT} . Thus,

$$H(s) = H_1(s) - e^{-sT} H_1(s) = \frac{1 - e^{-sT}}{sT} \quad (10)$$

Normalizing the complex frequency variable s through

$$\left. \begin{aligned} \gamma &= sT \\ H(s) &= F(\gamma) \\ H_1(s) &= F_1(\gamma) \end{aligned} \right\} \quad (10a)$$

we have

$$H(s) = F(\gamma) = \frac{1 - e^{-\gamma}}{\gamma} \quad (10b)$$

Strictly speaking, $H(s)$, being a transcendental function, is not realizable in terms of lumped circuit elements. We therefore have to settle for a rational approximation to it. Guillemin's approach avoids approximating e^{-sT} , introducing instead the approximation inherent in the truncation of a Fourier series. This is effected through the pair $h_2(t)$, $H_2(s)$ as follows: In analogy with Eq. (10) we have

$$F(\gamma) = H(s) = H_2(s) + e^{-sT} H_2(s) = (1 + e^{-\gamma}) F_2(\gamma) \quad (11)$$

Eliminating $e^{-\gamma}$ between Eqs. (10b) and (11), we obtain

$$F(\gamma) = \frac{2F_2(\gamma)}{1 + \gamma F_2(\gamma)} \quad (12)$$

To get $F_2(\gamma)$ we expand $h_2(t)$ in a Fourier series

$$h_2(t) = u(t) \sum_{k=1}^{\infty} b_k \sin\left(\pi k \frac{t}{T}\right) \quad (13)$$

where

$$b_k = \begin{cases} 0 & (k \text{ even}) \\ \frac{4}{\pi k T} & (k \text{ odd}) \end{cases} \quad (14)$$

Taking the Laplace transform of Eq. (13) we get

$$H_2(s) = \frac{\pi}{T} \sum_{k=1}^{\infty} \frac{k b_k}{s^2 + \left(\frac{\pi k}{T}\right)^2} \quad (15)$$

Finally, substitution of Eq. (14) in Eq. (15) yields

$$F_2(\gamma) = 4 \sum_{k=1}^{\infty} \frac{1}{\gamma^2 + [(2k-1)\pi]^2} \quad (16)$$

At this point we approximate by truncating the $F_2(\gamma)$ expansion after N terms. Distinguishing approximations by a circumflex we then have

$$\hat{H}_2(s) = \hat{F}_2(\gamma) = 4 \sum_{k=1}^N \frac{1}{\gamma^2 + [(2k-1)\pi]^2} \quad (17)$$

$$\hat{H}(s) = \hat{F}(\gamma) = \frac{2\hat{F}_2(\gamma)}{1 + \gamma \hat{F}_2(\gamma)} \quad (18)$$

We refer to the network realizing $\hat{H}(s)$ as the N th order network. As is evident from Eqs. (17) and (18), this transfer function has $2N$ poles.

III. Performance Criteria

$\hat{H}(s)$, the N th order approximation transfer function, approaches the ideal $H(s)$ as N tends to infinity. As we shall presently see, the number of components in the realization of the N th order filter is $3N - 1$. We are interested in the performance of $\hat{H}(s)$ as a function of N so that the improved performance associated with a more complex realization can be accurately gauged.

We start with a detailed examination of the effect of the approximation transfer function $\hat{H}(s)$ on the binary signal $x(t)$. In line with the preceding notation, we denote the output of this filter by $\hat{y}(t)$ and its δ -function response by $\hat{B}_T(t)$. Thus we have

$$\hat{y}(t) = \int_0^{\infty} x(\tau) \hat{B}_T(t - \tau) d\tau \quad (19)$$

Note that $x(\tau)$ may be expressed in terms of the box-car function as follows

$$x(\tau) = T \sum_{k=1}^{\infty} a_k B_T[\tau - (k-1)T] \quad (20)$$

Substituting in Eq. (19) we get

$$\hat{y}(t) = \sum_{k=1}^{\infty} a_k \hat{\mu}_T(t - kT) \quad (21)$$

where

$$\hat{\mu}_T(t) = T \int_0^T B_T(\tau) \hat{B}_T(\tau + t) d\tau \quad (22)$$

To see the significance of $\hat{\mu}_T(t)$, note that for the ideal filter $H(s)$, the above procedure yields

$$y(t) = \sum_{k=1}^{\infty} a_k \mu_T(t - kT) \quad (23)$$

with

$$\mu_T(t) = T \int_0^T B_T(\tau) B_T(\tau + t) d\tau \quad (24)$$

Unlike Eq. (22), the two functions here are identical and $\mu_T(t)$ is easily seen to be the simple triangle function displayed in Fig. 2. For arguments which are integral multiples of T , the effect of this function is very similar to that of a δ -function. Thus

$$\mu_T(kT) = \delta_{0k} \quad (25)$$

Hence

$$y(kT) = \sum_{i=1}^{\infty} a_i \mu_T[(k-i)T] = \sum_{i=1}^{\infty} a_i \delta_{ik} = a_k \quad (26)$$

The function of interest to us, $\hat{\mu}_T(t)$, is an approximation to $\mu_T(t)$. Thus, corresponding to Eq. (25), we have

$$\hat{\mu}_T(kT) = \hat{m}_k \quad (27)$$

with

$$\hat{m}_k \begin{cases} = 0 & (k < 0) \\ \approx 1 & (k = 0) \\ \approx 0 & (k > 0) \end{cases} \quad (28)$$

Note that we are considering here a family of functions, since $\hat{\mu}_T(t)$ is also a function of N . As N increases, $\hat{\mu}_T(t)$ approaches $\mu_T(t)$ and \hat{m}_k tends to δ_{0k} . This is evident in Fig. 3 which shows $\hat{\mu}_T(t - T)$ for $N = 3$.

Applying Eqs. (27) and (28) to Eq. (21) we get

$$\begin{aligned} \hat{y}(kT) &= \sum_{i=1}^k a_i \hat{m}_{k-i} = \sum_{i=0}^{k-1} a_{k-i} \hat{m}_i \\ &= \hat{m}_0 a_k + \sum_{i=1}^{k-1} a_{k-i} \hat{m}_i \end{aligned} \quad (29)$$

We recall now that our objective is to use $\hat{y}(kT)$ to determine a_k . Equation (29) shows that in addition to the

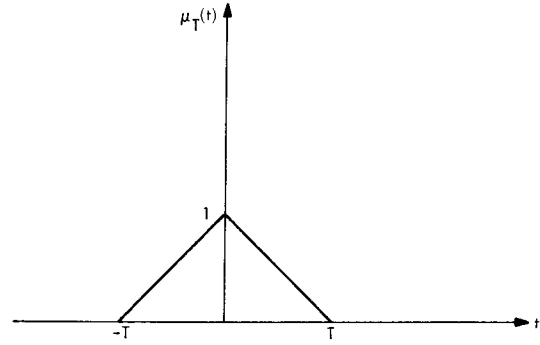


Fig. 2. Function $\mu_T(t)$

desired output, $\hat{m}_0 a_k$, we also get small contributions from all preceding bits. This raises the following question: Assuming that $a_k = 1$, what is the probability that application of the decision rule of Eq. (3) would yield $a_k = -1$? The sum in Eq. (29) is one contribution to this probability. The other major contribution is noise. Let us assume that the signal $x(t)$ feeding the filter is accompanied by white gaussian noise yielding the noise output $n(t)$. The filter output is now given by

$$z(t) = n(t) + \hat{y}(t) \quad (30)$$

In particular, for $a_k = 1$

$$z(kT) = \hat{m}_0 + \left\{ n(kT) + \sum_{i=1}^{k-1} \hat{m}_i a_{k-i} \right\} \quad (31)$$

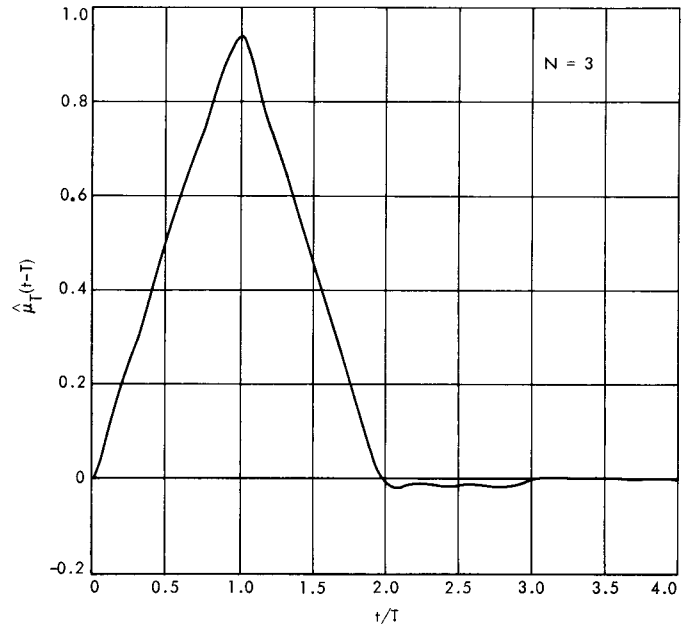


Fig. 3. Function $\hat{\mu}_T(t - T)$ for $N = 3$

Assuming now that the bits a_i are totally uncorrelated with themselves as well as with the noise samples, we get for the mean and variance of $z(kT)$

$$\left. \begin{aligned} E[z(kT)] &= \hat{m}_0 \\ \sigma_z^2 &= \sigma_n^2 + \sum_{i=1}^{k-1} \hat{m}_i^2 \end{aligned} \right\} \quad (32)$$

where σ_n^2 is the variance of the noise output. Finally, applying the law of large numbers we conclude that $z(kT)$ may be reasonably well approximated by a normally distributed random variable with the parameters prescribed in Eq. (32).

Application of the decision rule (3) to $z(kT)$ will yield the wrong answer ($a_k = -1$) when $z(kT) < 0$. Thus, the probability of error is given by the integral of the z distribution over negative z . Alternatively, using the normalized entity

$$Q(v) = \frac{1}{\sqrt{2\pi}} \int_v^\infty \exp\left(-\frac{t^2}{2}\right) dt \quad (33)$$

and denoting

$$\xi = \frac{\hat{m}_0}{\sigma_z} \quad (34)$$

we see that

$$\text{probability of error} = Q(\xi) \quad (35)$$

By symmetry, this is also the probability of error when $a_k = -1$.

The parameter ξ is a function of N through \hat{m}_0 as well as via σ_z . To compare the performance of networks of different orders, we need an explicit formulation of $\xi(N)$. To obtain this, let us assume that the input to the filter consists of the signal at levels of ± 1 V (across 1Ω) and an accompanying white noise with the (two-sided) spectral density v (in W/Hz). This means that

$$\sigma_n^2 = \int_{-\infty}^{\infty} |\hat{H}(i\omega)|^2 v df = v \int_0^\infty \hat{B}_T^2(t) dt \quad (\omega = 2\pi f) \quad (36)$$

It is convenient to express this in terms of the ideal network ($N = \infty$) for which

$$\sigma_n^2 = v \int_0^\infty B_T^2(t) dt = \frac{v}{T} \quad (37)$$

Thus we write for the network of order N

$$\sigma_n^2 = p_N \frac{v}{T} \quad (38)$$

where p_N is computable from the known $\hat{B}_T(t)$. It is independent of T and varies monotonically from $p_1 = 0.81$ to $p_\infty = 1$.

Applying Eqs. (38) and (32) to Eq. (34), we get

$$\xi^2 = \frac{\hat{m}_0^2}{p_N \frac{v}{T} + \sum_{i=1}^{k-1} \hat{m}_i^2} \quad (39)$$

Hence

$$v \equiv v_N = T \frac{\hat{m}_0^2}{p_N} \left[\frac{1}{\xi^2} - \sum_{i=1}^{k-1} \left(\frac{\hat{m}_i}{\hat{m}_0} \right)^2 \right] \quad (40)$$

For $N = \infty$ this simplifies to

$$v_\infty = \frac{T}{\xi^2} \quad (41)$$

Hence

$$\frac{v_N}{v_\infty} = \frac{\hat{m}_0^2}{p_N} \left[1 - \xi^2 \sum_{i=1}^{k-1} \left(\frac{\hat{m}_i}{\hat{m}_0} \right)^2 \right] \quad (42)$$

Equation (42) is an implicit formulation of the desired performance criterion. Given a prescribed error probability $Q(\xi)$, we find ξ from the relevant tables. Now an ideal network ($N = \infty$) would realize the prescribed performance with a noise power density v_∞ given by Eq. (41). If, however, we use an N th order network, we could still realize the prescribed $Q(\xi)$ provided the noise power density is reduced. Equation (42) computes this reduction.

Table 1 shows v_N/v_∞ (in dB) as a function of error probability for $1 \leq N \leq 7$.¹ The corresponding plots are shown in Fig. 4. Note that for an error probability of 10^{-3} , the simplest network ($N = 1$) is 0.69 dB worse than the ideal. The corresponding value for the second order network is 0.33 dB. From here on, however, the rate of improvement diminishes. Essentially the same pattern holds for all other values of the error probability in Fig. 4.

¹Actually the table values refer to the synthesized networks and thus reflect errors of the synthesis process. These, however, do not exceed 0.001 dB. See Section IV.

Table 1. v_N/v_∞ in dB, as a function of error probability and network order N

NETWORK ORDER	1	2	3	4	5	6	7
PROBABILITY OF ERROR							
1.-01	-.673	-.324	-.213	-.158	-.126	-.104	-.089
1.-02	-.681	-.327	-.214	-.159	-.126	-.105	-.089
1.-03	-.690	-.329	-.215	-.159	-.127	-.105	-.089
1.-04	-.699	-.332	-.216	-.160	-.127	-.105	-.089
1.-05	-.708	-.335	-.218	-.161	-.128	-.106	-.089
1.-06	-.717	-.338	-.219	-.162	-.128	-.106	-.089
1.-07	-.726	-.341	-.220	-.162	-.129	-.106	-.089
1.-08	-.736	-.344	-.222	-.163	-.129	-.107	-.089
1.-09	-.745	-.346	-.223	-.164	-.130	-.107	-.090
1.-10	-.754	-.349	-.224	-.165	-.130	-.107	-.090

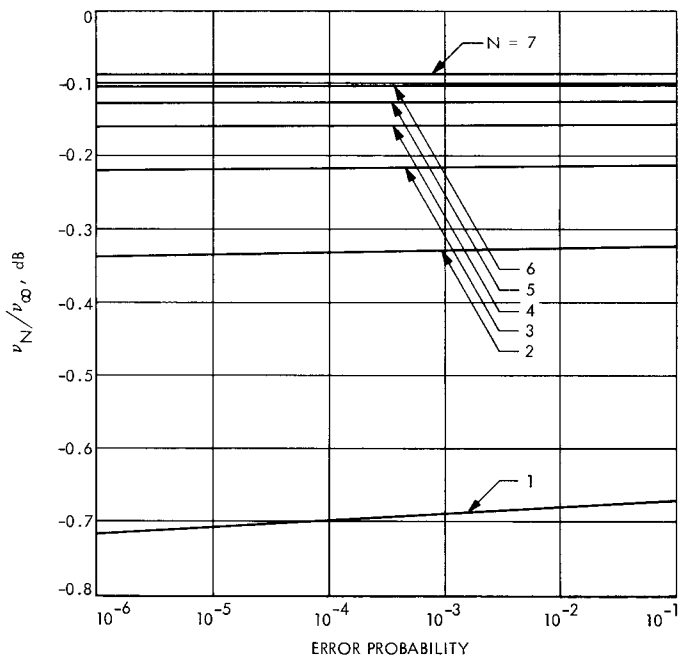


Fig. 4. v_N/v_∞ as a function of error probability and network order N

IV. The Filter Synthesis

Applying the criteria of the last section, we arrive at the specific $\hat{H}(s)$ that would satisfy the requirements of the problem at hand. It remains now to synthesize a network realizing this $\hat{H}(s)$.

The particular realization considered here is that of a lossless network inserted between a generator having a finite internal resistance and a purely resistive load. In other words, our goal is to design the lossless network indicated in Fig. 5 so as to realize

$$\frac{V_2(s)}{V_g(s)} = \alpha \hat{H}(s) = W(s) \quad (43)$$

The constant multiplier α is necessary to allow arbitrary values of

$$\rho = \frac{R_2}{R_1} \quad (44)$$

(See, however, the discussion associated with Eq. 60). The relationship between ρ and α can be established via the DC behavior of the filter. From Eqs. (17) and (18) we see that

$$\hat{H}(0) = 2\hat{H}_2(0) = \frac{8}{\pi^2} \cdot \sum_{k=1}^N (2k-1)^{-2} \neq 0 \quad (45)$$

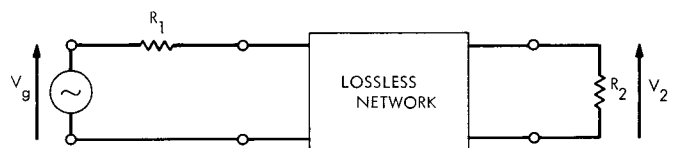


Fig. 5. Connection of the filter

But with the network being lossless and $\hat{H}(0) \neq 0$, Fig. 5 prescribes

$$\frac{V_2(0)}{V_g(0)} = \frac{R_2}{R_1 + R_2} = \frac{\rho}{1 + \rho} \quad (46)$$

Hence

$$\alpha = \frac{\rho}{\hat{H}(0)(1 + \rho)} \quad (47)$$

The synthesis considered will be that of a prototype filter with

$$\left. \begin{aligned} R_1 &= 1 \Omega \\ \rho &= 1 \\ T &= \pi \text{ sec} \end{aligned} \right\} \quad (48)$$

This particular choice of T allows some of the computations to be carried out with integers, leading to higher precision (see Eqs. 17 and 10a).

Filters realizing any other values of R_1 and T are trivially obtained from this prototype filter (see Eq. 65). Modification of ρ , however, calls for a new synthesis. Denoting

$$s^2 = q \quad (49)$$

we get for the prototype

$$\hat{H}_2(s) = \frac{4}{\pi^2} \sum_{k=1}^N \frac{1}{q + (2k-1)^2} = \frac{4}{\pi^2} \cdot \frac{A(q)}{B(q)} \quad (50)$$

where $A(q)$, $B(q)$ are polynomials of degrees $N-1$, N respectively. $B(q)$ is obtained directly by multiplying out its factors. The simplest way to obtain $A(q)$ is based on the observation that Eq. (50) implies that all residues of $A(q)/B(q)$ are 1. This leads to

$$A(q) = \frac{dB(q)}{dq} \quad (51)$$

Thus, for any finite N , $A(q)$, $B(q)$ are easily determined polynomials all of whose coefficients are expressible as sums of products of positive integers. Substituting Eq. (50) in Eq. (18), we express $W(s)$ directly in terms of these polynomials:

$$W(s) = \alpha \hat{H}(s) = \frac{\frac{2}{\pi} \alpha A(q)}{sA(q) + \frac{\pi}{4} B(q)} \quad (52)$$

The actual synthesis is straightforward and follows the standard insertion loss technique (Ref. 4). We are pointing here only highlights of the process.

The subsequent discussion will refer to the following three alternate matrix descriptions of the two port of Fig. 5:

$$\left. \begin{aligned} Z(s) &= \begin{bmatrix} Z_{11}(s) & Z_{12}(s) \\ Z_{21}(s) & Z_{22}(s) \end{bmatrix} \\ Y(s) &= \begin{bmatrix} Y_{11}(s) & Y_{12}(s) \\ Y_{21}(s) & Y_{22}(s) \end{bmatrix} \\ S(s) &= \begin{bmatrix} S_{11}(s) & S_{12}(s) \\ S_{21}(s) & S_{22}(s) \end{bmatrix} \end{aligned} \right\} \quad (53)$$

These are the open-circuit impedance matrix, short-circuit admittance matrix, and scattering matrix, respectively:

The two basic elements in the synthesis procedure are:

- (1) The transmission zeros of the network, that is, the zeros of $A(q)$ in our case.
- (2) Any one of $Z_{11}(s)$, $Z_{22}(s)$, $Y_{11}(s)$, $Y_{22}(s)$ ²

The transmission zeros are obtained directly from $A(q)$. The immittances mentioned in b are derived from $W(s)$ in an indirect way. First, a scattering matrix satisfying $W(s)$ is determined. The required immittances then follow the application of the standard transformations generating Z and Y from S .

In obtaining S , use is made of the fact that for a lossless two port and a transfer function whose numerator is an even s polynomial, the scattering matrix has the following form (Ref. 5):

$$S(s) = \frac{1}{g(s)} \begin{bmatrix} h(s) & f(s) \\ f(s) & -h_*(s) \end{bmatrix} \quad (54)$$

where g , h , f are real polynomials,³

$$h_*(s) = h(-s) \quad (55)$$

and

$$gg_* = hh_* + ff_* \quad (56)$$

²Strictly speaking, only $Z_{11}(s)$, $Y_{22}(s)$ are capable of generating all the network elements. Referring to Fig. 6, it is obvious that $Y_{11}(s)$ cannot determine C_N while $Z_{22}(s)$ will not determine L_1 .

³The polynomial $h(s)$ should not be confused with the time function $h(t)$ of Section I.

To realize $W(s)$ we identify

$$g(s) = sA(q) + \frac{\pi}{4} B(q) \quad (57)$$

$$f(s) = \psi A(q) \quad (58)$$

where

$$\psi = \frac{4\alpha}{\pi \sqrt{\rho}} = \frac{\frac{4}{\pi}}{\hat{H}(0)(\rho^{1/2} + \rho^{-1/2})} \quad (59)$$

To complete the determination of S , $h(s)$ is now obtained from Eq. (56). Note that this equation provides a unique determination only for the product hh_* . The determination of h itself is not unique. Generally, a given hh_* is consistent with a number of different h functions leading to different network realizations. In some situations, however, there is no real polynomial $h(s)$ corresponding to the prescribed hh_* . This happens to be the case for the first-order network ($N = 1$) where analysis shows a forbidden ρ range given by

$$\frac{1}{1.49} < \rho < 1.49 \quad (60)$$

In view of this, we modify the prototype specifications (48) as follows:

$$\left. \begin{aligned} R_1 &= 1 \Omega \\ \rho &\begin{cases} = 1.5, & N = 1 \\ = 1, & N > 1 \end{cases} \\ T &= \pi \text{ sec} \end{aligned} \right\} \quad (61)$$

Of the various possible $h(s)$, we examine here only those which satisfy the following constraint

$$\begin{aligned} &\text{All roots of } h(s) \text{ are on one side} \\ &\text{of the imaginary axis} \end{aligned} \quad (62)$$

Having determined S , we turn now to the transformation to Z and Y . This is most easily expressed in terms of the even and odd parts of g and h . Denoting

$$\left. \begin{aligned} g &= g_e + g_o & [g_e(-s) &= g_e(s); & g_o(-s) &= -g_o(s)] \\ h &= h_e + h_o & [h_e(-s) &= h_e(s); & h_o(-s) &= -h_o(s)] \end{aligned} \right\} \quad (63)$$

the required immittances are expressible as follows:

$$\left. \begin{aligned} \frac{Z_{11}}{R_1} &= \frac{g_e + h_e}{g_o - h_o} & Y_{11} R_1 &= \frac{g_e - h_e}{g_o + h_o} \\ \frac{Z_{22}}{R_2} &= \frac{g_e - h_e}{g_o - h_o} & Y_{22} R_2 &= \frac{g_e + h_e}{g_o + h_o} \end{aligned} \right\} \quad (64)$$

The actual synthesis applies the method of Fujisawa (Ref. 6) which yields a transformerless ladder configuration. We have chosen the synthesis realizing the transmission zeros as parallel resonance circuits as shown in Fig. 6. The combination of this configuration constraint and the roots constraint (62), limits the various possibilities so that there is only one prototype for each N value.⁴

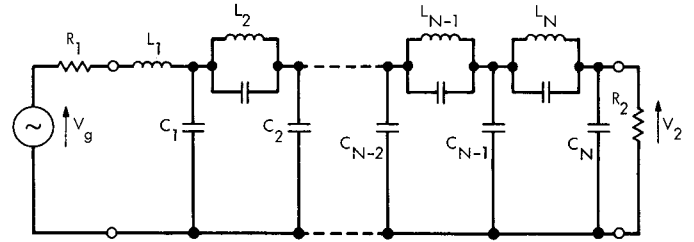


Fig. 6. Filter configuration

The computed parameter values of networks up to order 7 are shown in Table 1, which is directly related to Fig. 6, and is mostly self-explanatory ($R_{GEN} = R_1$; $R_{LOAD} = R_2$; capacitor $CR(I)$ resonates coil $L(I)$ at frequency $FR(I)$). A new entity introduced here is the "sample gain" which is defined as the ratio (in the absence of noise) of the filter output voltage at time kT to the generator voltage representing bit a_k .

To illustrate the use of this table assume that we wish to construct a matched filter for bit time T' and generator resistance R_1' . Assigning primes to all the parameters of this filter, we obtain them according to the following

⁴It should be borne in mind, though, that given a network such as that in Fig. 6, one could alter the connections, feeding the input in series with R_2 and taking the output across R_1 , realizing a transfer function which is a constant multiple of the original transfer function.

transformations applicable to inductances, capacities, and frequencies, respectively:

$$\left. \begin{aligned} \frac{L'}{L} &= \frac{T'}{T} \frac{R_1'}{R_1} \\ \frac{C'}{C} &= \frac{T'}{T} \frac{R_1'}{R_1} \\ \frac{f'}{f} &= \frac{T}{T'} \end{aligned} \right\} \quad (65)$$

With $T = \pi$ sec and $R_1 = 1 \Omega$, Table 2 provides all the information necessary to determine the "primed" filter.

A specific example is shown in Table 3 which refers to filters associated with a $10\text{-}\mu\text{s}$ bit time and a $50\text{-}\Omega$ generator.

V. Reliability of Element Values

To check the reliability of the computed element values we compute the performance of the synthesized network and compare it to the design goal. Since we are dealing here with a ladder configuration, it is relatively easy to obtain the transfer function of the synthesized network by multiplying out the ABCD matrices of the individual network sections. Note that this computation is quite precise since it involves the multiplication and summation of positive quantities only. Having thus obtained the "synthesized transfer functions" we compute from them v_N/v_∞ , the performance criteria described in Section III.

Comparing these to the values obtained from $\hat{H}(s)$ directly, we found that the discrepancies for all the networks shown in Table 2 do not exceed 0.001 dB. Furthermore, the values appearing in Table 1 are those of the synthesized networks so that even the above minor discrepancies should not be of concern to the user.

Table 2. Network elements for $T = \pi$ sec, $R_1 = 1 \Omega$

UNITS SECOND,HERTZ,CHM,HENRY,FARAD
T= 3.1415927+00
RGEN= 1.000+00

NET. ORDER	1	2	3	4	5	6	7
RLOAD	1.500+00	1.000+00	1.000+00	1.000+00	1.000+00	1.000+00	1.000+00
SAMPLE GAIN	6.170-01	5.076-01	5.051-01	5.039-01	5.031-01	5.033-01	0.000
L (1)	1.410+00	5.006-01	2.997-01	2.157-01	1.689-01	1.562-01	2.126-01
CR(1)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
C (1)	1.182+00	8.890-02	2.271-02	1.041-02	6.647-03	1.088-03	4.996-02
L (2)		4.908-01	7.719-02	2.869-02	1.459-02	9.807-03	-2.434-02
CR(2)		4.075-01	6.920-01	8.563-01	9.680-01	9.352-01	-2.645-01
FR(2)		3.559-01	3.418-01	3.354-01	3.318-01	3.295-01	3.278-01
C (2)		1.749+00	6.944-02	1.306-02	3.645-03	-4.937-02	8.556-02
L (3)			4.672-01	8.518-02	3.310-02	8.140-03	-1.746-02
CR(3)			4.642-01	6.573-01	7.667-01	1.777+00	-5.353-01
FR(3)			6.886-01	6.726-01	6.645-01	6.594-01	6.559-01
C (3)			1.995+00	7.011-02	1.335-02	3.031-02	-1.681-02
L (4)				4.597-01	8.799-02	2.798-02	-1.994-01
CR(4)				4.897-01	6.520-01	9.230-01	-7.354-02
FR(4)				1.015+00	9.991-01	9.903-01	9.846-01
C (4)				2.100+00	7.066-02	2.844-02	-3.459-02
L (5)					4.557-01	8.750-02	1.866-01
CR(5)					5.049-01	6.658-01	1.400-01
FR(5)					1.339+00	1.323+00	1.314+00
C (5)					2.160+00	7.378-02	-4.917-02
L (6)						4.525-01	1.130-01
CR(6)						5.157-01	5.209-01
FR(6)						1.662+00	1.646+00
C (6)						2.199+00	4.209-02
L (7)							4.587-01
CR(7)							5.138-01
FR(7)							1.983+00
C (7)							2.220+00

Table 3. Network elements for $T = 10 \mu s$, $R_1 = 50 \Omega$

UNITS MICROSECOND,MEGAHERTZ,OHM,MICROHENRY,MICROFARAD
 $T = 1.000000 \times 01$
 $RGEN = 5.000 \times 01$

NET. ORDER	1	2	3	4	5	6	7
RLOAD	7.500+01	5.000+01	5.000+01	5.000+01	5.000+01	5.000+01	5.000+01
SAMPLE GAIN	6.170-01	5.076-01	5.051-01	5.039-01	5.031-01	5.033-01	0.000
L (1)	2.244+02	7.967+01	4.769+01	3.434+01	2.688+01	2.487+01	3.384+01
CR(1)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
C (1)	7.526-02	5.660-03	1.446-03	6.624-04	4.232-04	6.925-05	3.181-03
L (2)		7.812+01	1.228+01	4.567+00	2.322+00	1.561+00	-3.874+00
CR(2)		2.594-02	4.405-02	5.452-02	6.162-02	5.954-02	-1.684-02
FR(2)		1.118-01	1.074-01	1.054-01	1.042-01	1.035-01	1.030-01
C (2)		1.114-01	4.421-03	8.317-04	2.321-04	-3.143-03	5.447-03
L (3)			7.435+01	1.356+01	5.268+00	1.296+00	-2.779+00
CR(3)			2.955-02	4.185-02	4.881-02	1.131-01	-3.408-02
FR(3)			2.163-01	2.113-01	2.087-01	2.072-01	2.061-01
C (3)			1.270-01	4.463-03	8.496-04	1.929-03	-1.070-03
L (4)				7.316+01	1.400+01	4.453+00	-3.173+01
CR(4)				3.117-02	4.151-02	5.876-02	-4.682-03
FR(4)				3.190-01	3.139-01	3.111-01	3.093-01
C (4)				1.337-01	4.498-03	1.811-03	-2.202-03
L (5)					7.252+01	1.393+01	2.970+01
CR(5)					3.214-02	4.239-02	8.915-03
FR(5)					4.208-01	4.158-01	4.129-01
C (5)					1.375-01	4.697-03	-3.130-03
L (6)						7.201+01	1.799+01
CR(6)						3.283-02	3.316-02
FR(6)						5.221-01	5.172-01
C (6)						1.400-01	2.679-03
L (7)							7.301+01
CR(7)							3.271-02
FR(7)							6.231-01
C (7)							1.413-01

VI. Concluding Remarks

The element values appearing in the example of Table 3 are quite reasonable. Thus, the main goal of synthesizing the matched filter has been achieved. However, if one is to apply this design in practice, various questions have to be answered, whether through computer simulation or with actually built filters. We briefly mention some of these here.

We start with the effects of stray capacities. Examination of Fig. 6 shows that most stray capacities could be incorporated into the circuit elements. An exception is the first section, that is, the section realizing the transmission zero at infinity. As the self-resonance of a physical coil

occurs at a finite frequency, we have here a factor which might be prominent in limiting the applicability of this design for very small T , i.e., very high symbol rates. Another unknown factor is the effect of losses in the elements, particularly in the coils.

Regarding the sensitivity of the performance criteria to deviations from the prescribed element values, we have some indirect evidence that as long as the transmission zero frequencies are adjusted to their correct values, the performance is relatively immune to slight deviations. Finally, it has been assumed at the outset that the sampling instants ($t = kT$) are precisely known. The effect of slight shifts in the sampling times merits further study.

References

1. Viterbi, A. J., *Principles of Coherent Communication*, Sect. 7.1. McGraw-Hill Book Co., Inc., 1966.
2. Meyer, R. A., "Filters with an Approximately Rectangular Impulse Response," *NTZ Commun. J.*, Vol. 5, No. 3, 1966.
3. Guillemin, E. A., *Synthesis of Passive Networks*. Sect. 6, Chapt. 15. John Wiley & Sons, 1957.
4. Saal, R., and Ulbrich, E., "On the Design of Filters by Synthesis," *IRE Trans. PGCT*, Vol. CT-5, No. 4, pp. 284-327, December 1958.
5. Belevitch, V., "Topics in the Design of Insertion Loss Filters," *IRE Trans. PGCT*, Vol. CT-2, No. 4, pp. 337-346, December 1955.
6. Fujisawa, T., "Realizability Theorem for Mid-Series or Mid-Shunt Low-pass Ladders Without Mutual Induction," *IRE Trans. PGCT*, Vol. CT-2, No. 4, pp. 320-325, December 1955.